USING *OLS* TO TEST FOR NORMALITY

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Abstract

Yitzhaki (1996) showed that the *OLS* estimator of the slope coefficient in a simple regression is a weighted average of the slopes delineated by adjacent observations. The weights depend only on the distribution of the independent variable. In this paper I demonstrate that equal weights can only be obtained if and only if the independent variable is normally distributed. This characteristic is used to develop a new test for normality which is distribution free and not sensitive to outliers. The test is compared with standard normality tests, in particular, the popular Jarque-Bera test. It is shown that the new test is a better power for testing normality against all classes of alternative distributions. Finally, the test is applied to check normality in time series data from major international financial markets.

Keywords : Regression weights, Jarque-Bera test, Kolmogorov-Smirnov test

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In a major paper, Yitzhaki (1996) presented the following important results regarding the Ordinary Least Squares (*OLS*) estimator of a simple regression coefficient:[1]

1. The *OLS* estimator of the slope coefficient is a weighted average of the slopes delineated by adjacent observations.
2. The weights used in averaging the slopes depend solely on the distribution of the observations of the independent variable.
3. In particular, if the independent variable is normally distributed, the weights are equal to the normal density. Hence, equal percentiles of the distribution receive equal weights for all the slopes of adjacent observations.

The main implication of Yitzhaki's results is that unless the independent variable is normally distributed, the *OLS* estimator is in fact a weighted regression estimator that attributes most of the weight to the more extreme observations. Usually, *OLS* is silent about the distribution of the independent variable, which is assumed to be non-random. Indeed, the issue of the distribution of the independent variable is not discussed in econometric texts. At most, normality of error terms is required as a prerequisite for statistical inference.

In a subsequent paper, Shalit & Yitzhaki (2002) showed that for observations characterized by fat tails such as financial data, so called "outliers" are receiving most of the explanatory power of the regression, thus yielding non-robust results. Removal of outliers is not a desirable practice as this eliminates valid information on the behavior of economic variables. Indeed, in light of the recent high volatility of security prices, extreme observations are the ones that contribute most in explaining price behavior. Hence, a major prerequisite for the practitioner is to test whether or not the data is normally distributed if one is to obtain robust results using *OLS*, independently of whether or not statistical inference is undertaken.

In this paper, I extend Yitzhaki's (1996) results by demonstrating that equal weights attributed to equal percentiles can only be obtained if and only if the independent variable is normally distributed. The main implication of this result is that it serves as the basis for a new test for normality without the need for specifying the sampling distribution. Indeed, since equal weights are necessary and sufficient for a distribution to be normal, the regression weights of

[1] The properties of the simple regression *OLS* estimator carry through to the multiple regression. See also Heckman, Urzua, and Vyitlacil (2006).

OLS are the only data needed to test for normality. The main advantage of the procedure is that it does not require categorizing the mean and the variance of the normal distribution for the null hypothesis and, therefore, it is robust and insensitive to sample outliers. Hence, this new test is entirely distribution free. In essence, the procedure consists of testing whether or not the weights used by *OLS* are equal. I devise the statistical test and compare its results with the performance of standard tests for normality, the most popular being the Jarque-Bera test (1980). I examine the new test by first exploring its size and its power and second, by using financial time-series data.

1. The *OLS* Regression Estimator

In the following, I summarize Yitzhaki's results claiming that the *OLS* regression estimator of the slope coefficient is a weighted average of the slopes of the lines defined by adjacent observations.

Consider a simple regression model where variables are continuously random with a joint density function $f(X, Y)$, and where X is the independent variable and Y is the dependent variable. Let f_X, F_X, μ_X, and σ^2_X denote the marginal density, the marginal cumulative distribution, the expected value and the variance of X. Assume the existence of first and second moments.

Theorem 1 (**Yitzhaki, 1996**): *Let $E(Y|X) = \alpha + \beta X$ be the best linear predictor of Y, given X. Then β_{OLS} is the weighted average of the slopes of the regression curve $g(x)= E(Y|X=x)$, namely:*

$$\beta_{OLS} = \int_X w(x)\,\delta(x)\,dx \tag{1}$$

where $\delta(x)= g'(x)$ and $w(x) > 0$, $\int w(x)dx = 1$ and the weights are given as:

$$\begin{aligned} w(x) &= (1/\sigma_X^2)\,[\,\mu_X F_X(x) - \int_{-\infty}^{x} t\, f_X(t)\,dt\,] \\ &= \int_{-\infty}^{x} (\mu_X - t)\, f_X(t)\,dt \,/\, \sigma_X^2 \ . \end{aligned} \tag{2}$$

Proof: See Appendix A

Theorem 1 presents the *OLS* regression coefficient as a weighted average of the dependent variable differences, conditional on the independent variable differences. The weighting scheme depends solely upon the cumulative distribution of the independent variable.

In particular, for the normal distribution with expected value μ_X and variance σ^2_X, the weighting scheme from Equation (2) becomes (Yitzhaki, 1996):

$$w(x) = -\frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x} t e^{-(t-\mu_X)^2/2\sigma_X^2}\, dt = \frac{1}{\sqrt{2\pi\,\sigma_X^2}} e^{-(x-\mu_X)^2/2\sigma_X^2} \qquad (3)$$

The weights equal the density function of the normal distribution of *X*. Hence, equal percentiles of the distribution receive equal weights and the explanatory power of the regression is distributed evenly among the observations.

2. The Normal Distribution of the Independent Variable

In this section, I demonstrate that the only valid occurrence for which equal weights are ensured is when the independent variable of the *OLS* regression is normally distributed. For any other probability distribution of the explanatory variable, we obtain uneven weights, implying that some observations provide higher explanatory power and others less. Henceforth, robust *OLS* results can be obtained solely for a normally distributed independent variable.

Theorem 2: *The weights from Equation* (2) *equal the probability density function if and only if the independent variable is normally distributed.*

Proof [2]: In Equation (2) I substitute $w(x) = f_X(x)$ for all x and obtain a differential equation that is solved by integrating by parts to yield:

$$\sigma_X^2\ f(x) = (\mu_X - x)\, F_X(x) + \int_{-\infty}^{x} F_X(t)\, dt \qquad (4)$$

where $dF(x) = f(x) \equiv F'(x)$. If one differentiates Equation (4) with respect to x one obtains:

$$\sigma_X^2\ F_X''(x) - (\mu_X - x)\, F_X'(x) = 0 \qquad (5)$$

The general solution to the differential Equation (5) is given as:

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[2] In this proof, I consider probability distributions that are absolutely continuous with respect to the Lebesgue measure. Another proof by Preminger and Shalit (1999) using central moments is shown in Appendix B.

$$F_X(x) = C \int_{-\infty}^{x} e^{-(t^2 - 2\mu_X t)/2\sigma_X^2} dt\,, \tag{6}$$

where the constant C is found for $F(\infty) = 1$ as:

$$C = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\mu_X^2/2\sigma_X^2}\,. \tag{7}$$

After substituting for C one obtains the solution for Equation (5) which is the normal probability distribution function:

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x} e^{-(t-\mu_X)^2/2\sigma_X^2} dt\,. \tag{8}$$

Q.E.D.

3 Testing for Normality

3.1 The Test

From Theorem 2 we are able to derive a new test for normality for any set of observations of a random variable. The test is entirely distribution free as it depends only on the weights expressed by Equation (2). Indeed if all the weights are equal, the random variable is normally distributed.

The test is composed of two steps. The first step consists of calculating the weights, and the second tests whether the weights are equal. I consider the sample X_i , i=1, …, n and test whether it is drawn from a normal distribution.

In the first step, I rank the sample in increasing order, $x_{i+1} \ge x_i \ge x_{i-1}$, $i = 2, \ldots, n-1$. An estimator for the weights $w(x)$ is obtained by substituting:

(i) the sample average $\bar{X}_n$ for the mean μ_X ,

(ii) the sample mean up to observation i $\bar{X}_i = \sum_{j=1}^{i} X_j / i$ for the conditional mean $\mu_X(x) = \int_{-\infty}^{x} t\, f_X(t)\, dt$,

(iii) the sum of squares S_X^2 for the variance σ_X^2 , and

(iv) the relative rank $\frac{i}{n}$ for the cumulative distribution function $F_X(x)$.

The weight of the segment $\Delta x_i = x_{i+1} - x_i$ then becomes:

$$w_i = \frac{i}{n}\left[\frac{\bar{X}_n - \bar{X}_i}{S_X^2}\right]\Delta x_i \quad . \tag{9}$$

From the ranked sample, I compute the weights for n-1 segments to examine whether these weights are even. This is the second step where I test whether the weights follow a uniform distribution. For this reason, I propose to use the standard Kolmogorov-Smirnov (KS) test on the cumulative probability distribution of the uniform distribution. Let the empirical distribution be the cumulative sum of the weights $F_w(i) = \sum_{j=1}^{i} w_j$ computed from Equation (9) and let the theoretical distribution be $F_e(i) = \frac{i}{n-1}$. The KS statistic is obtained as

$$KS = \max_i \left| F_w(i) - F_e(i) \right| \tag{10}$$

A possible alternative for the significance points for the KS statistic can be the standard critical values that are provided in Table1. Another is the significance values obtained by sizing the sample as done below.

To illustrate the entire procedure, I present in Table 2 the results of the test for 50 observations that were drawn from a gamma distribution.[3] The weights are calculated in the fifth column of the table. In the ninth column, the absolute difference between $F_w(i)$ and $F_e(i)$ is shown. The KS statistic for this sample is 0.221774 which exceeds the critical values for $\alpha = .1$ and $\alpha = .05$ rejecting the null hypothesis that the sample is drawn from a normal distribution. The Jarque-Bera (JB) statistic for this sample is 5.58184, which also rejects normality for $\alpha = .1$ and $\alpha = .05$.

3.2 The Size of the Test

To calculate the significance points for the test, a Monte Carlo simulation is performed for a normally distributed set of observations drawn by using a random generator. For the simulation, I first use 100,000 and then 500,000 repetitions. For each replication, N observations are created using a normal random number generator with mean 5 and standard deviation 5. The estimates of the 1%, 5% and 10% significance points of the test are shown in Table 3a for 100,000 and in Table 3b for 500,000 replications. There are no considerable differences between the two tables.

[3] The observations are taken from Madanski (1988), Table 4.

3.3 The Power of the Test

To compare the test with other normality tests I investigate its power under various distribution alternatives, again using Monte Carlo simulations. The results are reported in Table 4 for 100,000 and 500,000 replications. The other normality tests include the standard Jarque-Bera (J-B) test, the Lagrange multiplier test by Deb and Sefton (LM (DS)) which is a J-B test with new calculated critical values, and the Lilliefors test which is a modified Kolmogorov-Smirnov test. All the simulations and alternative tests were run in MATLAB and are available from the author. The distributions used to investigate the power are: Normal (10,25), Gamma (3,1), Student t (5), Beta (3,2), Exponential(10), Chi-squared (5), and Weibull (3,1).

Looking at Table 4, we see that the new test has the best power for testing normality against most of the alternative distributions, mainly for the smaller numbers of observations. This result is quite promising when one compares the performance of the new test against the two J-B benchmark tests. This result especially holds when the alternative is the Gamma, the Beta, the Exponential, the Chi-squared and the Weibull distributions. However, the two J-B tests have a better power when the alternative is the Student-t distribution. When the number of replications increases to 500,000 and the number of observations is greater than 100 the power of the new test is not as great as one would expect for a test for normality.

3.4 The Jarque-Bera test, its validity, and its modifications

Although most of the professional statisticians preconize the use of the Shapiro-Wilk test as well its improvements by Shapiro and Francia (1972) and by Royston (1982), the Jarque-Bera test remains to be the most popular test for normality in the field of economics and finance. This is because it is easy to calculate and is a combination of two moments, namely, skewness and kurtosis. The Jarque-Bera (1980) statistic is also known as the D'Agostino-Pearson (1973) and the Bowman-Shelton (1975) test. As such, it has become a standard feature in many econometric packages. The standard JB statistic is given by

$$JB = \frac{N}{6}\left\{\left[\sqrt{b_1}\right]^2 + \frac{1}{4}\left[b_2 - 3\right]^2\right\} \tag{11}$$

where $\sqrt{b_1}$ is the sample skewness and b_2 is the sample kurtosis, which, under the null hypothesis of normal distribution, are asymptotically independent and normally distributed. Hence, the J-B statistic follows a χ^2 distribution with 2 degrees of freedom.

Of course, the J-B test is not without its skeptics. The validity of the J-B test has been questioned as its asymptotic distribution is problematic and its convergence is very slow, both of which result in an undersized test. Various alternative versions have been suggested by Deb and Sefton (1996), Poitras (2006), Gel and Gastwirth (2008), but none have succeeded in supplanting the J-B test. In the present paper, I did not evaluate the new test against the previous standard normality tests since they were compared to the popular J-B test for the purpose of devising modifications. From Tables 3 and 4 one can assert the marginal superiority of the new test relative to the J-B test. Based on our results, we can conclude that not only is the size of the proposed test comparable to the J-B test, but the new test also has a better power. It was suggested by a reader that the standard tests for normality calculate parameters that might conceal non-normality in the aggregation process. The new test is based on each couple of observations, which may be the source of its statistical power.

4 Applying the Test to Financial Time-Series

The empirical part of the paper aims to try the test on financial time-series. Normality of returns is a theoretical prerequisite for the use of the mean-variance model in portfolio analysis as well for the Capital Assets Pricing Model (*CAPM*). For this reason, I test the normality of returns on 5 US financial markets indices and 5 European financial markets indices using both monthly returns and daily returns.

The US indices that are used include: the Dow-Jones Industrial Average (DJIA), the Standard and Poor's 500 (S&P500), the Standard and Poor's 100 (S&P100), the Nasdaq composite Index (NASDQ), and the Russell 2000(RUSS2000). The European financial indices are: the London Financial Times Shares Exchange (FTSE), the Frankfurt German stock market index (DAX), the Paris French Stock Market index (CAC40), the Zurich Swiss Market Index (SMI) and the Amsterdam Dutch Stock Exchange Index (AEX). For the monthly data, I collected 120 returns from October 1999 until October 2009. For the daily data, I collected 250 daily returns from October 7, 2008 until October 2, 2009.

The results are displayed in Table 5. For most of the monthly series, the J-B statistic follows the new OLS statistic. According to the J-B test and the OLS test, normality is rejected for most of the monthly financial returns. However, the test results for the RUSSELL2000 index, the S&P100 index and the NASDAQ Composite index are ambiguous. Indeed, following the

OLS weight test, normality is not rejected for the NASDAQ and the RUSSELL2000 whereas the J-B test does not reject normality for the NASDAQ and the S&P100 indices.

5 Conclusion and Implications

It was shown that if and only if the independent variable of an OLS regression is normally distributed, the regression weights attributed to the observations are equal to the density of the normal distribution. This implies that normality is a necessary and sufficient condition for each percentile of the population to receive an equal share of the weights used by the regression. Only then, will all observations contribute evenly to the *OLS* estimation and yield robust estimators. This supports Shalit & Yitzhaki's (2002) claim that if observations are not normally distributed, other regression techniques should be used to ensure robustness.

This result on regression weights s allowed us to derive a new test for normality for any sample of observations. The proposed test is an equality test using the *KS* procedure when the alternative distribution is a uniform distribution. Hence besides calculating the regression weights, the only parameter that is required to estimate the expected distribution is the number of observations. The simulation results showed that, compared to standard tests for normality, the new test has the best power for testing normality against most of the classes of alternative continuous distributions.

Appendix A

Proof of Theorem 1 (Yitzhaki, 1996)

Let E(Y|X) = α + β X be the best linear predictor of Y, given X. Then β_{OLS} is the weighted average of the slopes of the regression curve g(x)= E(Y|X=x), namely:

$$\beta_{OLS} = \int_X w(x)\,\delta(x)\,dx \tag{12}$$

where δ(x)= g′(x) and w(x) > 0, ∫w(x)dx = 1 and the weights are given as:

$$\begin{aligned} w(x) &= (1/\sigma_X^2)\,[\,\mu_X F_X(x) - \int_{-\infty}^{x} t\, f_X(t)\,dt] \\ &= \int_{-\infty}^{x} (\mu_X - t)\, f_X(t)\,dt \,/\, \sigma_X^2 \;. \end{aligned} \tag{13}$$

Proof: As $\beta = \mathrm{cov}(Y,X)/\sigma_X^2$, one expresses the numerator as:

$$\mathrm{cov}(Y,X) = E_X E_Y[(X-\mu_X)Y] = E_X(X-\mu_X)E_Y(Y \mid X=x) = \int (x-\mu_X)g(x)f_x(x)dx\,,$$

where $g(x) = E_Y(Y \mid X = x)$ is the conditional expectation of Y given X. Integrating by parts with $V(x) = \int_{\infty}^{x}(t-\mu_X)f_X(t)dt$, $V'(t) = (t-\mu_X)f_X(t)$, $U(x) = g(x)$, and $U'(x) = g'(x)$, leads to

$$\mathrm{cov}(Y,X) = [\int_{-\infty}^{x}(t-\mu_X)f_X(t)dt]g(x)\,|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\int_{-\infty}^{x}(\mu_X - t)f_X(t)dt]g'(x)dx\,.$$

As second moments exist, the first term converges to zero. Hence,

$$\mathrm{cov}(Y,X) = \int_{-\infty}^{\infty}[\mu_X F_X(x) - \int_{-\infty}^{x} tf_X(t)dt]\;g'(x)dx\,.$$

The sum of weights equals one since the same procedure can be applied to the denominator σ_X^2 which equals $\sigma_X^2 = \int_{-\infty}^{\infty}[\mu_X F_X(x) - \int_{-\infty}^{x} tf_X(t)dt]\,dx$.

Q.E.D.

Appendix B: Alternative proof of Theorem 2 (Preminger and Shalit, 1999)

We first show that:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{x} (\mu - t)\ f(t)\, dt\ \ (x - \mu)\ \ dx = \tfrac{1}{2} \int_{-\infty}^{\infty} (x - \mu)^3\ f(x)\, dx \tag{14}$$

This is obtained by integrating Equation (13) by parts and recalling that:

$$\frac{1}{2}(x^2 - 2\mu\, x)\ \int_{-\infty}^{x} (\mu - t)\ f(t) dt \Big|_{-\infty}^{+\infty} = 0 \tag{15}$$

Starting with symmetric distributions, we know that for these distributions the odd central moments vanish. Thus:

$$\int_{-\infty}^{\infty} (\ x - \mu\ )^3\ f\ (\ x\ )\, dx = 0 \tag{16}$$

Dividing Equation (13) by σ_X^2 and using Equation (15), we obtain:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{x} [\ (\mu - t)\ f\ (t) / \sigma_X^2\ ]\, dt\, (x - \mu)\, dx = 0 \tag{17}$$

Inserting the weights expressed by Equation (2) we obtain:

$$\int_{-\infty}^{\infty} w(x)\, (x - \mu)\, dx = \int_{-\infty}^{\infty} [\ w(x) - f(x)\ ]\, x\, dx = 0 \tag{18}$$

Hence, for some symmetric distributions $w(\ x\ ) = f\ (\ x\ )$ for all $x \neq 0$. However, for asymmetric distributions, the weights *cannot* equal the density function because Equation (13) does not equal zero. Yitzhaki (1996) proved that if the distribution is normal the weights are equal to the density.

We now show that if the distribution is symmetric but not normal Equation (17) cannot hold. With the same technique used for obtaining Equation (13) we demonstrate that:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{x} (\ \mu - t)\ f(t)\, dt\ \ (x - \mu)^2\, dx = \tfrac{1}{3} \int_{-\infty}^{+\infty} (x - \mu\ )^4\ f\ (x)\, dx \tag{19}$$

The right-hand side of Equation(13) is related to the kurtosis of a distribution that is defined as:

$$\gamma_4 = \int_{-\infty}^{\infty} (x - \mu_X\ \ )^4\ \ f_X\, (\ x\ ) dx / \sigma_X^4 - 3 \tag{20}$$

For all non-normal distributions, the kurtosis is non-zero. In Equation (18) we replace the right-hand-side integral with γ_4 and divide it by σ_X^2 to obtain:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{x} [\,(\mu - t)\ f\ (t\,)\,/\,\sigma^2]\,dt\ (x - \mu)^2\,dx - \sigma^2 = \frac{\gamma_4}{3}\,\sigma^2 \tag{21}$$

We now insert the weights defined by Equation (2) to obtain:

$$\int_{-\infty}^{+\infty} [\,w\,(x) - f\,(x)\,]\,(x - \mu)^2\,dx = \frac{\gamma_4}{3}\sigma^2 \tag{22}$$

For symmetric distributions with a non-null kurtosis, Equation (21) cannot be zero. Hence, for these distributions, the weights cannot equal the density function. Q.E.D.

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Table 1: Kolmogorov-Smirnov Critical Values

Percentages	.10	.05	.01	.005
Critical values	$1.22/\sqrt{n}$	$1.36/\sqrt{n}$	$1.63/\sqrt{n}$	$1.731/\sqrt{n}$

Table 2: Test Procedure for 50 Observations Drawn from a Gamma Distributed Sample

Variate X	Rank i	$\Delta X = X_{i+1} - X_i$	$E_i(X) = \sum_{j=1}^{i} X_j / i$	w_i Weight	$F_w(i) = \sum_{j=1}^{i} w_j$	$f_e(i) = 1/n\text{-}1$	$F_e(i) = \sum_{j=1}^{i} f_e(j)$	$\lvert F_w - F_e \rvert$
6.48875	1	1.21929	6.48875	0.00383	0.00383	0.020408	0.020408	0.016579
7.70804	2	0.75346	7.098395	0.004577	0.008407	0.020408	0.040816	0.032409
8.4615	3	4.29524	7.552763	0.038148	0.046555	0.020408	0.061224	0.01467
12.75674	4	0.32381	8.853758	0.003549	0.050103	0.020408	0.081633	0.031529
13.08055	5	0.35226	9.699116	0.004573	0.054677	0.020408	0.102041	0.047364
13.43281	6	0.506	10.3214	0.007563	0.062239	0.020408	0.122449	0.06021
13.93881	7	1.09636	10.83817	0.018445	0.080684	0.020408	0.142857	0.062173
15.03517	8	0.40584	11.3628	0.007514	0.088198	0.020408	0.163265	0.075067
15.44101	9	0.31254	11.81593	0.006294	0.094492	0.020408	0.183673	0.089181
15.75355	10	0.03146	12.20969	0.000683	0.095175	0.020408	0.204082	0.108907
15.78501	11	0.87417	12.53472	0.020344	0.115519	0.020408	0.22449	0.10897
16.65918	12	0.05509	12.87843	0.00136	0.116879	0.020408	0.244898	0.128019
16.71427	13	0.71471	13.17349	0.018651	0.13553	0.020408	0.265306	0.129776
17.42898	14	2.19712	13.47746	0.060161	0.195691	0.020408	0.285714	0.090023
19.6261	15	0.16341	13.88736	0.004624	0.200315	0.020408	0.306122	0.105807
19.78951	16	0.1705	14.25625	0.004975	0.20529	0.020408	0.326531	0.12124
19.96001	17	0.08532	14.59176	0.002563	0.207853	0.020408	0.346939	0.139086
20.04533	18	0.10388	14.89474	0.003208	0.211061	0.020408	0.367347	0.156286
20.14921	19	0.47839	15.17129	0.015167	0.226228	0.020408	0.387755	0.161527
20.6276	20	0.37236	15.44411	0.012082	0.23831	0.020408	0.408163	0.169854
20.99996	21	0.00935	15.70867	0.00031	0.238619	0.020408	0.428571	0.189952
21.00931	22	1.32929	15.94961	0.044938	0.283558	0.020408	0.44898	0.165422
22.3386	23	0.53382	16.22739	0.018288	0.301846	0.020408	0.469388	0.167542
22.87242	24	0.06833	16.50427	0.002366	0.304212	0.020408	0.489796	0.185584
22.94075	25	0.29846	16.76173	0.010438	0.31465	0.020408	0.510204	0.195554
23.23921	26	0.57502	17.01086	0.020283	0.334933	0.020408	0.530612	0.195679
23.81423	27	0.33701	17.26284	0.011956	0.34689	0.020408	0.55102	0.204131
24.15124	28	0.10757	17.50885	0.003832	0.350722	0.020408	0.571429	0.220707
24.25881	29	0.98166	17.74161	0.035095	0.385817	0.020408	0.591837	0.20602
25.24047	30	0.13033	17.99157	0.004654	0.390471	0.020408	0.612245	0.221774
25.3708	31	1.11016	18.22961	0.039579	0.43005	0.020408	0.632653	0.202603
26.48096	32	0.51117	18.48747	0.018096	0.448146	0.020408	0.653061	0.204915
26.99213	33	1.29893	18.74518	0.045549	0.493695	0.020408	0.673469	0.179775
28.29106	34	0.90931	19.02594	0.031381	0.525075	0.020408	0.693878	0.168802
29.20037	35	0.36193	19.31664	0.012233	0.537308	0.020408	0.714286	0.176977
29.5623	36	0.7441	19.60124	0.024576	0.561885	0.020408	0.734694	0.172809
30.3064	37	0.26785	19.89057	0.008606	0.570491	0.020408	0.755102	0.184611
30.57425	38	1.27841	20.17172	0.03987	0.610362	0.020408	0.77551	0.165149
31.85266	39	0.853	20.47123	0.025613	0.635975	0.020408	0.795918	0.159943
32.70566	40	0.60638	20.77709	0.017417	0.653392	0.020408	0.816327	0.162934
33.31204	41	0.03803	21.08282	0.001039	0.654431	0.020408	0.836735	0.182304
33.35007	42	0.08208	21.3749	0.002126	0.656557	0.020408	0.857143	0.200586
33.43215	43	0.61991	21.6553	0.015172	0.671729	0.020408	0.877551	0.205823
34.05206	44	6.4137	21.93705	0.147137	0.818865	0.020408	0.897959	0.079094
40.46576	45	5.33323	22.3488	0.108374	0.927239	0.020408	0.918367	0.008872
45.79899	46	0.73737	22.85858	0.012385	0.939624	0.020408	0.938776	0.000848
46.53636	47	0.55198	23.36236	0.007256	0.94688	0.020408	0.959184	0.012303
47.08834	48	5.54644	23.85666	0.052151	0.999031	0.020408	0.979592	0.019439
52.63478	49	0.20533	24.44396	0.000969	1	0.020408	1	3.55E-15
52.84011	50							

Table 3a Critical Values for the Test Obtained Using 100,000 Repetitions

Observations	α=0.10	α=0.05	α=0.01
20	0.2736	0.3093	0.3801
50	0.1816	0.2037	0.2485
100	0.1308	0.147	0.1784
200	0.0937	0.1051	0.1277
500	0.0599	0.067	0.0813

Table 3b Critical Values for the Test Obtained Using 500,000 Repetitions

Observations	α=0.10	α=0.05	α=0.01
20	0.2741	0.309	0.3795
50	0.1815	0.2039	0.2486
100	0.1308	0.1468	0.1784
200	0.0938	0.1051	0.1278
500			

Table 4a Powers of Normality Tests for 100,000 Repetitions

α=5%		DISTRIBUTIONS					
Observations	**Test**	**N(10,25)**	**Γ(3,1)**	**t(5)**	**β(3,2)**	**χ(5)**	**Wei(3,1)**
20	OLS	0.0502	0.34657	0.18266	0.05512	0.40563	0.75294
	J-B	0.05026	0.30584	0.23017	0.0215	0.34908	0.62921
	LM(DS)	0.05079	0.30781	0.23132	0.02186	0.35125	0.6319
	Lilliefors	0.05028	0.2273	0.13267	0.06581	0.26887	0.58394
50	OLS	0.05091	0.76726	0.29015	0.11872	0.83342	0.99494
	J-B	0.05014	0.67199	0.4327	0.01835	0.74085	0.97612
	LM(DS)	0.04962	0.66974	0.43141	0.01787	0.7386	0.97553
	Lilliefors	0.05047	0.51567	0.20988	0.11946	0.59545	0.96173
100	OLS	0.05952	0.98115	0.43336	0.31544	0.99278	1
	J-B	0.05027	0.95953	0.64611	0.09385	0.98128	1
	LM(DS)	0.0502	0.64593	0.64593	0.09322	0.98122	1
	Lilliefors	0.05034	0.82726	0.33234	0.23226	0.89284	0.99996
200	OLS	0.05011	0.99994	0.59413	0.64086	1	1
	J-B	0.05023	0.99993	0.8656	0.70679	0.99999	0.1
	LM(DS)	0.04973	0.99992	0.86493	0.69965	0.99999	1
	Lilliefors	0.04927	0.98895	0.54009	0.4844	0.99713	1

α =10%		DISTRIBUTIONS					
Observations	**Test**	**N(10,25)**	**Γ(3,1)**	**t(5)**	**β(3,2)**	**χ(5)**	**Wei(3,1)**
20	OLS	0.10141	0.47058	0.26119	0.11575	0.52992	0.8461
	J-B	0.09992	0.44892	0.31386	0.06373	0.50249	0.80128
	LM(DS)	0.101	0.45136	0.31543	0.06494	0.50525	0.80437
	Lilliefors	0.10135	0.33806	0.20876	0.12705	0.38599	0.70825
50	OLS	0.09938	0.85406	0.37853	0.21462	0.90211	0.99836
	J-B	0.10044	0.83731	0.51949	0.13145	0.88917	0.99761
	LM(DS)	0.10018	0.83685	0.51917	0.13069	0.88872	0.99757
	Lilliefors	0.10136	0.64759	0.30527	0.21005	0.71871	0.98419
100	OLS	0.10069	0.99078	0.51305	0.43456	0.9967	1
	J-B	0.10133	0.99148	0.71758	0.47046	0.99724	1
	LM(DS)	0.10142	0.99152	0.71767	0.47125	0.99724	1
	Lilliefors	0.09928	0.90369	0.4476	0.36033	0.94575	1
200	OLS	0.10119	0.99998	0.70236	0.79491	1	1
	J-B	0.10034	1	0.90235	0.93675	1	1
	LM(DS)	0.09966	1	0.90196	0.93562	1	1
	Lilliefors	0.09969	0.99651	0.66085	0.64065	0.99921	1

Table 4b **Powers of Normality Tests for 500,000 Repetitions**

α=5%		DISTRIBUTIONS						
Observations	**Test**	**N(10,25)**	**Γ(3,1)**	**t(5)**	**β(3,2)**	**Exp(10)**	**χ(5)**	**Wei(3,1)**
20	OLS	0.0499	0.3518	0.1824	0.0548	0.7512	0.4055	0.7504
	J-B	0.0497	0.3072	0.2304	0.0215	0.6285	0.3504	0.6275
	LM(DS)	0.0502	0.3091	0.2316	0.0218	0.6310	0.3525	0.6300
	Lilliefors	0.0493	0.2294	0.1322	0.0647	0.5824	0.2673	0.5815
50	OLS	0.0494	0.7671	0.2886	0.1186	0.9951	0.8344	0.9951
	J-B	0.0506	0.6734	0.4304	0.0195	0.9755	0.7430	0.9756
	LM(DS)	0.0498	0.6704	0.4290	0.0178	0.9757	0.7409	0.9758
	Lilliefors	0.0498	0.5146	0.2117	0.1190	0.9615	0.5933	0.9613
100	OLS	0.0493	0.9770	0.4113	0.2803	1.0000	0.9906	1.0000
	J-B	0.0503	0.9579	0.6460	0.1034	1.0000	0.9806	1.0000
	LM(DS)	0.0498	0.9589	0.6459	0.0951	1.0000	0.9814	1.0000
	Lilliefors	0.0498	0.8254	0.3309	0.2325	0.9999	0.8927	0.9999
200	OLS	0.0498	0.9999	0.5942	0.6425	1.0000	1.0000	1.0000
	J-B	0.0507	0.9999	0.8662	0.7026	1.0000	1.0000	1.0000
	LM(DS)	0.0497	0.9999	0.8658	0.7023	1.0000	1.0000	1.0000
	Lilliefors	0.0499	0.9885	0.5413	0.4860	1.0000	0.9970	1.0000

α =10%		DISTRIBUTIONS						
Observations	**Test**	**N(10,25)**	**Γ(3,1)**	**t(5)**	**β(3,2)**	**Exp(10)**	**χ(5)**	**Wei(3,1)**
20	OLS	0.0996	0.4718	0.2583	0.1128	0.8427	0.5288	0.8426
	J-B	0.0993	0.4510	0.3136	0.0631	0.7998	0.5032	0.7992
	LM(DS)	0.1003	0.4539	0.3152	0.0643	0.8026	0.5063	0.8021
	Lilliefors	0.0998	0.3397	0.2074	0.1263	0.7070	0.3849	0.7063
50	OLS	0.0992	0.8538	0.3781	0.2165	9.9984	0.9036	0.9984
	J-B	0.1008	0.8359	0.5171	0.1336	0.9974	0.8899	0.9974
	LM(DS)	0.1005	0.8360	0.5171	0.1303	0.9976	0.8901	0.9975
	Lilliefors	0.0999	0.6465	0.3072	0.2095	0.9840	0.7188	0.9840
100	OLS	0.0997	0.9906	0.5134	0.4363	1.0000	0.9967	1.0000
	J-B	0.1002	0.9912	0.7171	0.4699	1.0000	0.9970	1.0000
	LM(DS)	0.0998	0.9915	0.7174	0.4717	1.0000	0.9972	1.0000
	Lilliefors	0.1001	0.9023	0.4454	0.3630	1.0000	0.9462	1.0000
200	OLS	0.1000	1.0000	0.7013	0.7942	1.0000	1.0000	1.0000
	J-B	0.1006	1.0000	0.9032	0.9353	1.0000	1.0000	1.0000
	LM(DS)	0.0995	1.0000	0.9029	0.9359	1.0000	1.0000	1.0000
	Lilliefors	0.0999	0.9963	0.6620	0.6418	1.0000	0.9993	1.0000

Table 5: OLS Normality Test and J-B Normality Test Statistics for Monthly and Daily Selected Financial Series

INDICES	DJIA	SP500	SP100	RU2K	NASDAQ	FTSE	DAX	CAC40	SSMI	AEX
Monthly(120 Obs)										
OLS	0.147634	0.181817	0.173951	0.118089	0.112189	0.189191	0.132139	0.149822	0.214265	0.229718
JB	7.90626	10.7069	3.93096	5.24512	1.90627	10.1059	15.0046	5.01005	7.50581	19.0402

INDICES	DJIA	SP500	SP100	RU2K	NASDAQ	FTSE	DAX	CAC40	SSMI	AEX
Daily (250 Obs)										
OLS	0.156797	0.132325	0.147046	0.070527	0.124226	0.152465	0.165372	0.16304	0.184832	0.174608
JB	142.577	89.1529	112.52	8.79569	55.8693	134.308	157.465	148.254	248.808	99.8212